The Tinctures and Implicit Quantification Over Worlds

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> … one must keep a bright lookout for unintended and unexpected changes thereby brought about in the relations of different significant parts of the diagram to one another. Such operations upon diagrams, whether external or imaginary, take the place of the experiments upon real things that one performs in chemical and physical research. Chemists have ere now, I need not say, described experimentation as the putting of questions to Nature. Just so, experiments upon diagrams are questions put to the Nature of the relations concerned (4.530).[1]

The diagrammatic nature of mathematical reasoning suggests that as my power to create diagrams increases, so too will my capacity for fruitful mathematical reasoning. Peirce's own work involved an unending series of experiments with different diagrammatic notations, all interesting, some difficult, some extremely fruitful. And the diagrammatic notations available are not only a function of some kind of "internal mental activity." As Dewey has noted, "Breathing is an affair of the air as truly as of the lungs; digesting an affair of food as truly as of tissues of stomach" (Dewey, 15); so analogously is mathematical reasoning an affair of the diagrams available as truly as of the "mind" (which is then not limited to something inside the head, but includes the relevant diagrams, "external" as well as "internal"); so does mathematical reasoning have its "alembics and cucurbits" just as surely as does chemistry. In doing mathematical reasoning, we make of the diagrams "instruments of thought," and advances in the technology of diagrams can directly affect our patterns of reasoning. I can imagine Peirce spending hours (and dollars) in a modern artists' supply store, marveling at the rainbow of pencils and pens of all types and the diagrammatic possibilities that this technology opens up. And what would he think of the contemporary computer? A handy tool for writing a paper such as this, or for quickly doing all sorts of calculations, indeed. But ***also*** a medium for doing ***logic***, for facilitating the examination and understanding of the process of mathematical reasoning.

Without going into great technical detail (about computers and programming), I will sketch out an approach to Peircean logic employing the graphical capabilities of the modern Personal Computer. Only a limited graphics capability is required—essentially, just the garden variety VGA; for better video resolution, certain supersets of VGA are appropriate, but all we assume is the 18-bit RGB, 256 color scheme native to the VGA. I have done a considerable amount of programming in this environment and have found that, indeed, the diagrams involved are fruitful in presenting "unintended and unexpected changes thereby brought about in the relations of different significant parts of the diagram to one another."

In "Peirce and Philo" (1992), I point out that Peirce viewed the *de inesse* conditional as an instantiation of an integrating abstraction which he called the *hypothetical*; he held that his development of notation for quantification was necessary to treat adequately of the hypothetical, and as I note, in 1902 he remarks that

> the quantified subject of a hypothetical proposition is a *possibility*, or *possible case*, or *possible state of things*. In its primitive state, that which is *possible* is a hypothesis which in a given state of information is not known, and cannot certainly be inferred, to be false. The assumed state of information may be the actual state of the speaker, or it may be a state of greater or less information. Thus arise various kinds of possibility (2.347).

In the context of **EG**, quantifications are handled by transformations involving the

Line of Identity (**LI**); **LI**'s are *Implicitly Quantified Variables* (Zeman 1964, 1967). The signs Peirce calls *selectives* are also implicitly quantified variables; a selective (usually a letter of the alphabet, but possibly some other sign) may be attached to a hook (of a "spot," or predicate; a hook is the graphical equivalent of a blank in a predicate) or to the end of a **LI**; the set of identical selectives in a given graph is treated as the set of identical quantified variables in the scope of their common binding quantifier in standard algebraic logic. Thus that set of selectives is an alternative notation for a ligature connecting the points to which the selectives are attached, and any "Rules for Selectives" would be derived from the appropriate rules for **LI**'s.

We find Peirce (in 4.518 ff.) introducing a peculiar type of selective in one of his discussions of modality in a graphical context. Although it would be fascinating to follow him through the development of this notion, we will reserve this for another occasion. Suffice to say for now that he here treats states of information as values for selectives (*i.e*., for *quantified variables*), and even develops (4.521) a ligature or concatenation of **LI**'s to serve as such (implicitly) quantified variables. He also presents a sign "to express that a state of information **B** follows after a state of information **A**" (4.522); this is cut from the same cloth as is 2.347 and other places where he deals with the hypothetical. So ***states of information*** can be represented by selectives attached to hooks (which would have to be of a peculiar kind) associated with graphs; the effect of attaching a selective ***x*** to such a hook on a graph ***g*** is to assert that "There is a state of Information ***x*** such that ***g*** (or the proposition expressed by ***g***) holds at ***x***."

Peirce is here, of course, very close to the contemporary idea of possible-world semantics. I propose in this paper to develop this Peircean idea in a natural Peircean context.

The notion of "Selective for a state of information" will be given another representation, one that is, in fact, employed by Peirce, although not fully developed in some of the directions we will go here. In the "Prolegomena to an Apology for Pragmaticism" (***PAP***—4.530-72), Peirce explores a diagrammatic notation which, among other things, will enable us to deal with "states of things" in a variety of modalities:

> When any representation of a state of things consisting in the applicability of a given description to an individual or limited set of individuals otherwise indesignate is scribed, the Mode of Tincture of the province on which it is scribed shows whether the Mode of Being which is to be affirmatively or negatively attributed to the state of things described is to be that of Possibility, when Color will be used; or that of Intention, indicated by Fur; or that of Actuality shown by Metal. Special understandings may determine special tinctures to refer to special varieties of the three genera of Modality. Finally, the Mode of Tincture of the March may determine whether the Entire Graph is to be understood as Interrogative, Imperative, or Indicative (4.554).

My suggestion is that a tincture in the sense of ***PAP*** be taken as a *selective* for a state of information, or a state of things, or a state of affairs. This would mean that tincture in a broader sense will be a line of identity for states of affairs, much as is displayed in 4.522 and 523; the scribing of a graph on an area belonging to a tincture would be like attaching that "peculiar kind of hook" to the **LI** which is the tincture.[2] Whether thought of as a selective or as a line of identity, then, the tincture would be an implicitly quantified variable which draws its values from a domain of states of affairs, or of ***Possible Worlds***! In ordinary algebraic modal semantics, we commonly make use of a binary predicate which takes statements as one argument and terms or term-variables for "possible worlds" as the other; if we take "**@**' as such a relation, then (corresponding to use of the

hook to which attach selectives or ligatures for states of affairs) the algebraic

(1) $$\exists \boldsymbol{x}(\alpha @ \boldsymbol{x})$$

will be read "Proposition α is true at some world x"; we would accomplish the same assertion in this version of "Tinctured Existential Graphs" by scribing α on an appropriate area of the tincture corresponding to the variable ***x***.[3]

A tincture has, literally, two sides, called by Peirce the *recto* and the *verso* of that tincture. Based on this fact, as we shall see, we interpret a tincture in the diagrammatic system of VGA256 graphics to be a pair of colors $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ where ***x*** and ***y*** are either color indexes or are **RGB** color representations.[4] Peirce comments that

> Should the Graphist desire to negative a Graph, he must scribe it on the *verso*, and then, before delivery to the Interpreter, must make an incision, called a *Cut*, through the Sheet all the way round the Graph-instance to be denied, and must then turn over the excised piece, so as to *expose* its rougher surface carrying the negatived Graph-instance (4.556).

This would be represented in more ordinary notation by

(2) $$\sim\exists \boldsymbol{x}(\alpha @ \boldsymbol{x})$$

which is the same as

(3) $$\forall \boldsymbol{x} \sim(\alpha @ \boldsymbol{x})$$

Although in ***PAP*** Peirce tells us that "We are to imagine that the Graphist always finds Provinces[5] where he needs them" (4.553), he is not tremendously clear about how we are to interpret and work with these Provinces (which are thought of as overlying each other) in the most general case. *Cuts* as discussed in 4.556, however, do give us one pretty clear picture of access to other tinctures. These cuts may be thought of as penetrating down through the current sheet and into an underlying Province, and so permitting us to "get at" new tinctures.

A tincture is a variable with a structure: as we have indicated, we think of it as having a *recto* and a *verso*. These "sides" draw their values from the *recto-verso* pairs of individual **SA**'s, each of which represents a "state of affairs," or "possible world." We have mentioned that in the context of VGA256, each tincture may be thought of as a pair <***R*,*V***>, whose members may be either VGA256 color indexes or **RGB** color representations. Where ***R*** and ***V*** are color indexes (0-255), we will take them as *bit-wise (logical) complements* of each other with respect to 255 (= 0xFF, = 1111 1111 binary). When they are **RGB** colors drawn from among the 64^3 = 262,144 colors available in this mode, the pair associated with the tincture may also be thought of as

$$<<\mathbf{r}_{\mathrm{R}},\mathbf{g}_{R},\mathbf{b}_{R}>,<\mathbf{r}_{V},\mathbf{g}_{V},\mathbf{b}_{V}>>.$$

The triples making up this pair are the red, green, and blue values for the recto and verso respectively.[6] Our understanding is that these are color complements of each other (in the sense that $\mathbf{r}_{\mathrm{V}} = 63\text{-}\mathbf{r}_{\mathrm{R}}$, and so also for green and blue), so that the recto-verso pair are complementary colors (as <red,cyan>, <blue,yellow>, etc.). A basic map of the colors upon which we will draw in the present discussion is shown in Figure 1.[7]

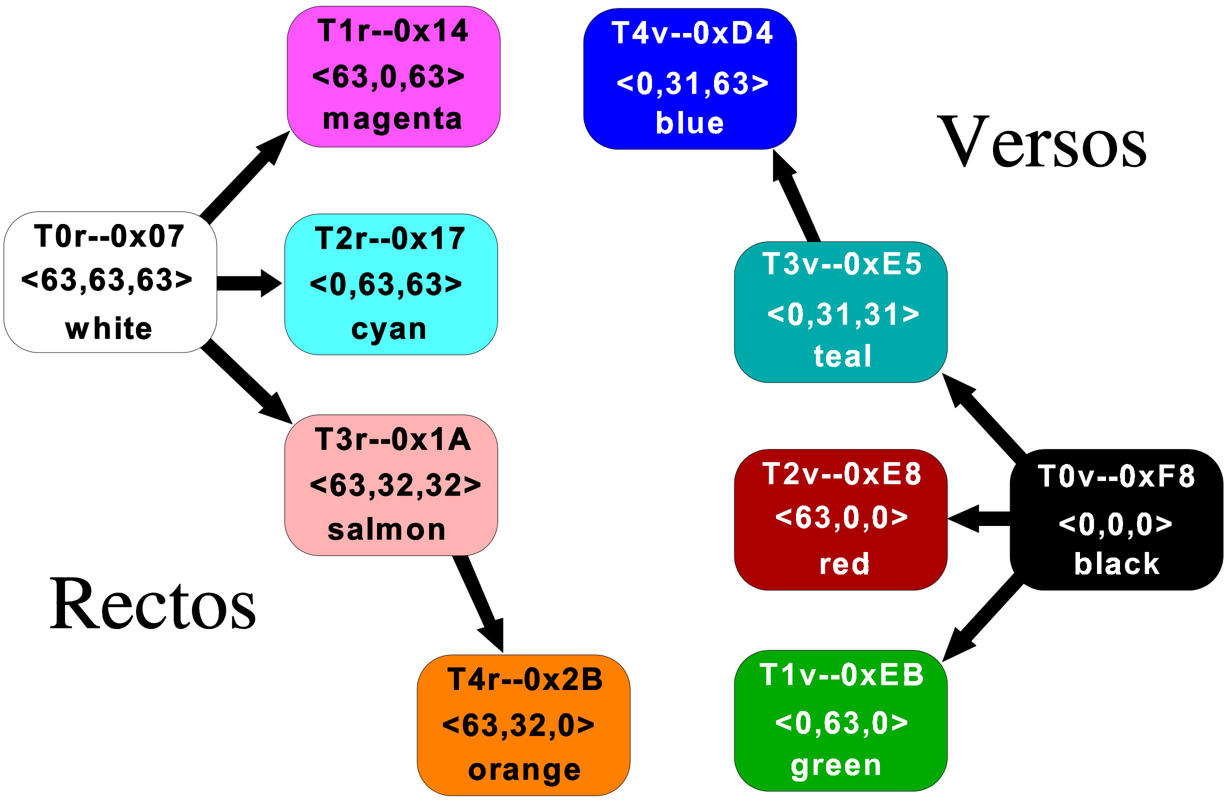

Figure 1

This diagram designates rectos and versos for five tinctures, which will be known respectively as **Tincture 0**, **Tincture 1**, etc; the tinctures shown are for present purposes only; many other patterns are possible. In terms that Peirce uses in ***PAP***, we might think of our **Tincture 0** as a *metal* ("*Argent*," most likely) and the other four as *colors* (4.553).

The relationship between *metals* and *colors*—indeed, between tinctures in general—is left vague in ***PAP***; the arrows in Figure 1 display a fairly natural relationship between tinctures which we will describe at some length later. Now, Peirce discusses tinctures and their placement as follows:

> Every part of the exposed surface shall be tinctured … The whole of any continuous part of the exposed surface in one tincture shall be termed a Province. The border of the sheet has one tincture all round; and we may

> imagine that it was chosen … in agreement between the Graphist and the Interpreter at the outset. The province of the border may be called the March. Provinces adjacent to the March are to be regarded as overlying it. Provinces adjacent to those Provinces, but not to the March, are to be regarded as overlying the provinces adjacent to the March, and so on. We are to imagine that the Graphist always finds Provinces where he needs them. (4.553).

Peirce is not, unfortunately, too explicit about how the various "Provinces" are bounded, nor about how we relate different provinces to each other. The interpretation of "tincture" we are suggesting will tie the operations on tinctures to the rules for **LI**'s and "ligatures" in general. Let us at present work with a limited manner of dealing with tinctures; though other ways of approaching the matter may be developed, the approach I will suggest is a start, and is consistent with the way that Peirce approached modality and "possible worlds." In a paper from the same year (1906) as ***PAP***, Peirce discusses an "Improvement to the Gamma Graphs"; which "improvement gives substantially, as far as I can see, nearly the whole of that Gamma part which I have been endeavoring to discern" (4.578). He is employing colors here, in a slightly different manner than in ***PAP***:

> In working with Existential Graphs, we use, or at any rate imagine that we use, a sheet of paper of different tints on its two sides. Let us say that the side we call the *recto* is cream white while the *verso* is usually of somewhat bluish grey, but may be of yellow or of a rose tint or of green (4.573).

The notational improvement he has in mind includes this color difference, so that when we perform a cut in the Gamma context,

> the cut may be imagined to extend down to one or another depth into the paper, so that the overturning of the piece cut out may expose one stratum or another, these being distinguished by their tints; the different tints representing

> different kinds of possibility (4.578).

This echoes remarks of Peirce's in the 1903 ***Lowell Lectures***. There, in trying to describe the relationship between "actual existent fact" and "possibility," he remarks that

> although the universe of existential fact can only be conceived as mapped upon a surface by each point of the surface representing a vast expanse of fact, yet we can conceive the facts [as] sufficiently separated upon the map for all our purposes; and in the same sense the entire universe of logical possibilities might be conceived to be mapped upon a surface. Nevertheless, if we are going to represent to our minds the relation between the universe of possibilities and the universe of actual existent facts, if we are going to think of the latter as a surface, we must think of the former as three-dimensional space in which any surface would represent all the facts that might exist in one existential universe (4.514).

The "strata" of possibility in 4.578 are continuous with each other, but may be approximated by the pages of a book, or a stack of sheets. This analogy may also be applied to the tinctures of ***PAP***; the cuts of the logic we are working with are then cuts indeed, going down to some appropriate level of paper in the book, "always find[ing] Provinces where [we] need them." For our present purposes, then, we will access tinctures by cutting down to them, and overturning the little stack of papers thus freed; the bottommost of the layers cut is the one we are accessing "through the cut," and so what is displayed in the cut, once the stack of paper thus freed is overturned, is the *verso* of that accessed layer. Tinctures other than the one on top (very likely, the ***Alpha-Beta*** Sheet of Assertion, representing the "actual existent universe") are then seen only within cuts—this extends recursively to any area of the graphs involved; passage from one tincture to another will always be via a cut, from the *recto* of one tincture to the *verso* of the other, or vice-versa.

This dictates much of what can happen with the rules of inference for ligatures when they are applied to tinctures; in particular, it affects the rules for iteration and deiteration. Note that when a graph containing **LI**'s is iterated, an exact copy of the graph is scribed in the same area as the original, or in an area enclosed by at least the same cuts as the original; this is just the same as the alpha rule of iteration-deiteration. An additional permission is that **LI**'s which are *unenclosed in* the graph being iterated may be attached by ligature to the copies of those same **LI**'s in the graph-instance resulting from the iteration; for example, as in Figure 2.

Figure 2

Both the transformation shown and its inverse are permitted by this rule. Note that if a ligature is *enclosed in* the graph being iterated, that graph may be iterated only in simple alpha style; else the two graphs in Figure 3 would both give rise by iteration to Figure 4:

Figure 3

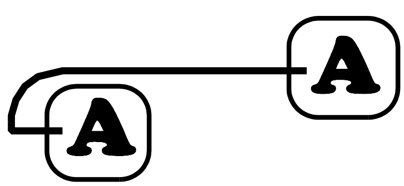

Figure 4

This would suggest that the two graphs of Figure 3 are equivalent, which they are not.

Figure 5

Although alpha iteration of the right graph in Figure 3 results in the two identical graphs of Figure 5, when we interpret the translation (into standard logical notation) of the graph including these two, the situation is complicated by the fact that **LI**'s must be translated using quantifiers and variables:

(1) $$\sim\exists \boldsymbol{x} \boldsymbol{A} \boldsymbol{x} \wedge \sim\exists \boldsymbol{x} \boldsymbol{A} \boldsymbol{x}$$

But, although the $\boldsymbol{x}$'s in the above are alphabetically the same, they are discontinuous with each other (as are the **LI**'s in Figure 5) this is emphasized by the fact that

(2) $$\sim\exists \boldsymbol{x} \boldsymbol{A} \boldsymbol{x} \wedge \sim\exists \boldsymbol{y} \boldsymbol{A} \boldsymbol{y}$$

is logically equivalent to (1).

Now, we have noted that *new* tinctures will be accessed only via cuts; this means that if I am working on an area on the recto of **Tincture 0**, I would encounter another tincture (say **Tincture 1**) as follows; **Tincture 1** is here what we will call *isolated*, which is to say that it does not occur on the sheet of assertion outside of the shown enclosure:

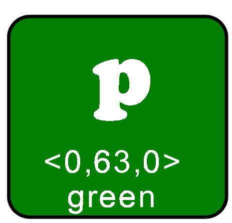

Figure 6

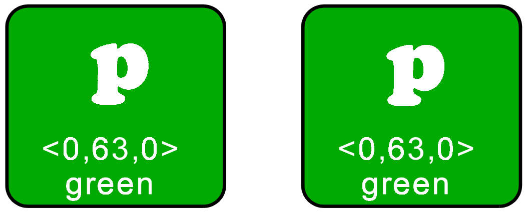

Figure 7

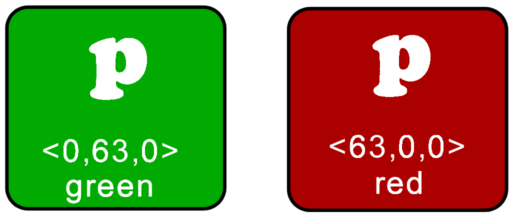

Figure 8

If I apply alpha iteration to the graph of Figure 6, the result would **not** be Figure 7. This last graph would be the analog in tinctures of Figure 4; Figure 6 is the analog in tinctures of the **right** graph in Figure 3. The identification of quantified variables implicit in the sameness of color in both sub-graphs of Figure 7 could not result from an iteration of Figure 6. The proper result of such an iteration is Figure 8,where the tincture represented here by **red** (the *verso* of **Tincture 2**) is entirely new to the total graph (if there are any isolated tinctures (***other than*** **Tincture 1**) "hidden" in **p**, they must be similarly changed in the copy of the graph iterated).

The iteration rules for tinctures, based on the beta rules, may be stated as follows:

$\mathbf{Itr}_{\gamma}$: A partial graph may be iterated in the same area or inward. *Isolated tinctures* in the original will be changed in the copy to *new tinctures accessible*[8] *to the tincture of the area to which the iteration is performed.*

$\mathbf{Dit}_{\gamma}$: A partial graph which might have resulted from $\mathbf{Itr}_{\gamma}$ may be deiterated.

Although movement from Figure 6 to Figure 7 would represent illegal manipulations of the **LI** which is **Tincture 1**, there are times when maintaining or making color identifications of areas will be perfectly ok. Before we examine some such, however, let us consider some interpretations of graphs involving tinctures. As a first such example, consider the graph in Figure 6. It starts on the *recto* of **Tincture 0**, which we may think of as the "ordinary" **SA**. Reading inward, we cross a cut into the *verso* of **Tincture 1**. The tincture there is a selective for "possible worlds," and the formula $\boldsymbol{p}$ is scribed on that *verso*. The effect is the same as

(3) $$\sim \exists w(\boldsymbol{p} \mathbin{@} \boldsymbol{w})$$

This is strongly suggestive of certain interpretations of modal propositions in standard modal semantics; also, Peirce's own intended use of the tinctures (especially of those he called "colors") is to express modality; various interpretations and rules of transformation which have been suggested for the tinctures (see Roberts 1974, Zeman 1974) would, in fact, result in reading the graph in Figure 6 as "$\boldsymbol{p}$ is impossible," or

(4) $$\sim \boldsymbol{Mp}$$

Although this is attractive, and is indeed the direction we are going here, it is not yet supported by formula (3). The standard "relational" semantics[9] requires an "accessibility relation" between possible worlds; say $\boldsymbol{Rxy}$, which would intuitively read "world $\boldsymbol{x}$ has access to world $\boldsymbol{y}$." Intuitively, of course, "access" can mean a great variety of things, depending on the kind of modality we are dealing with. Once we have the relation $\boldsymbol{R}$, we can extend our use of the predicate @ to include modal functions:

$$(\boldsymbol{L}\alpha) \mathbin{@} \boldsymbol{b} \text{ iff } \forall \boldsymbol{w}(\boldsymbol{Rbw} \supset \alpha \mathbin{@} \boldsymbol{w}), \text{ and}$$

$$(\boldsymbol{M}\alpha) \mathbin{@} \boldsymbol{b} \text{ iff } \exists \boldsymbol{w}(\boldsymbol{Rbw} \wedge \alpha \mathbin{@} \boldsymbol{w}).$$

The colors (from VGA256) which we employ here give us a ready-made path to the accessibility relation. Suppose that we have two tinctures (worlds), $\boldsymbol{u}$ and $\boldsymbol{v}$, associated respectively with the *recto-verso* pairs $<\boldsymbol{a},\boldsymbol{b}>$ and $<\boldsymbol{c},\boldsymbol{d}>$. With color $\boldsymbol{h}$ (in terms of red, green, blue) as

<$\mathbf{r}_h$,$\mathbf{g}_h$,$\mathbf{b}_h$>, we say that (colors) $\boldsymbol{e} \leq \boldsymbol{f}$ iff $\mathbf{r}_e \leq \mathbf{r}_f$ ***and*** $\mathbf{g}_e \leq \mathbf{g}_f$ ***and*** $\mathbf{b}_e \leq \mathbf{b}_f$; we might say in this case that $\boldsymbol{e}$ is a *sub-color* of $\boldsymbol{f}$. Then (with $\boldsymbol{u}$ as <$\boldsymbol{a}$,$\boldsymbol{b}$> and $\boldsymbol{v}$ as <$\boldsymbol{c}$,$\boldsymbol{d}$>):

> $\boldsymbol{Ruv}$ iff $\boldsymbol{c} \leq \boldsymbol{a}$ (equivalently, $\boldsymbol{b} \leq \boldsymbol{d}$, by the relationship between colors representing *rectos* and *versos*)

This is not the only possible meaning of $\boldsymbol{R}$, of course, but is a good natural one for these initial efforts. Loosely speaking, then, "*recto*-colors" have access to their "sub-colors," and "*verso*-colors" are accessed by their sub-colors. In interpreting a tinctured graph we will take the relation $\boldsymbol{R}$ to be asserted for a given tincture at the outermost point at which that tincture appears; this tincture will thus be accessed by the one immediately enclosing it. This is, of course, in addition to accesses which follow from the properties of $\boldsymbol{R}$.

For present purposes, we begin our graphical-logical work on the basic (alpha-beta) **SA**, whose tincture is **Tincture 0** (in **PAP** terms, *Argent*).[10] This tincture[11] is the "base," representing the "possible world" in which our graph-work is thought of as beginning. Access to **Tincture 0** is obtained on the very first *recto*—its own—which we encounter (unlike access to the other worlds we deal with in this presentation,[12] which we find only by cutting down to them and overturning the resultant stack of sheet-pieces). Considering **Tincture 0** to be a selective (or part of a **LI**), its presence, even when blank, is a claim that "There is a world, a universe of discourse, in which certain (as yet) indefinite propositions are true." Following the procedures used, say, in semantic tableaux for quantifiers, we may assign a name to this world—let us call it "$\boldsymbol{a}$"—provided that name has not been used before (and so designates an entirely indefinite[13] world). Given this name, the graph of Figure 6 would now have the interpretation

(3') $$\sim\exists \boldsymbol{w}(\boldsymbol{Raw} \wedge (\boldsymbol{p} \mathbin{@} \boldsymbol{w}))$$

which would be equivalent to asserting $\sim\boldsymbol{Mp}$ in the world represented by **Tincture 0**, that is, world $\boldsymbol{a}$. $\boldsymbol{Lp}$ would be represented (as holding in the *Argent* world) in these terms by Figure

9.

Magenta is the *recto* of the same tincture whose *verso* is green; since the cut within

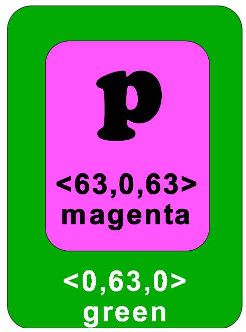

Figure 9

which the magenta appears is within the area of the green, there is no problem with tincture-identity. Thought of as a Line of Identity,[14] **Tincture 1** simply extends inward across the green-to-magenta cut, and the quantification implicit in it is determined by the position of its outermost (green) portion. This tincture is accessed, as we have noted, by **Tincture 0**. With $\boldsymbol{a}$ as the (indefinite) name of **Tincture 0**, we have, as the ordinary logic interpretation of Figure 9:

(5) $$\sim\exists \boldsymbol{w}(\boldsymbol{Raw} \wedge \sim(\boldsymbol{p} \ @\ \boldsymbol{w}))$$

This, of course, is equivalent to

(5') $$\forall \boldsymbol{w}(\boldsymbol{Raw} \supset \boldsymbol{p} \ @\ \boldsymbol{w})$$

which is equivalent to asserting $\boldsymbol{Lp}$ in the *argent* world.

Figure 10

In what follows, we shall perform some graphical transformations (deductions) involving tinctures, and as we go we shall introduce appropriate rules for tinctured gamma graphs based on the alpha and beta rules. As with most such graphical deductions, our first step will be a simple use of rule **Dc**$_\alpha$, giving Figure 10.

At this point we may introduce variants of the double-cut rules which will apply in the tinctured graphs. Since a tincture is a **LI** or selective, **Tincture 1** in Figure 9 is like a **LI** with its outermost portion in the space between the two cuts in that figure. By the Beta rules, then, those two cuts might not be placed or removed as a double cut. In Figure 10, however, **Tincture 0** begins outside the outermost cut and continues through the annular space into the innermost cut; considered as a **LI**, it would not prohibit the making or erasing of the double cut. Based on the Beta rules, the double-cut rules for tinctures are:

Dc+$_\gamma$: A pair of cuts with nothing between them but blank **SA**, and with the same tincture outside the outer cut, between the two cuts, and within the inner may be scribed around any graph (***Rule of Positive Double Cut for Tinctures***).

Dc-$_\gamma$: A pair of cuts with nothing between them but blank **SA**, and with the same tincture outside the outer cut, between the two cuts, and within the inner may be erased from around any graph (***Rule of Negative Double Cut for Tinctures***).

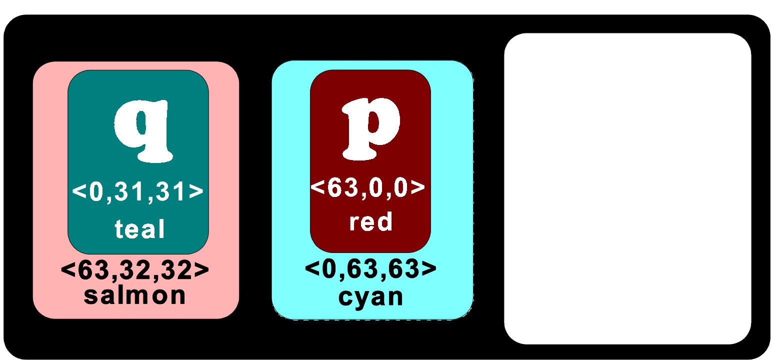

Figure 11

The next step will to be to use the (Alpha) rule of insertion in odd to obtain Figure 11. This rule is unaltered by the tinctures. An application of the rule of Iteration appropriate to tinctures is now required. The partial graph containing ***q*** in Figure 11 is of **Tincture 3**. This tincture is *isolated* in this graph—its outermost occurrence is just where we see it. By the reasoning which led to Figure 8, an iteration of this graph must result in a replica with an entirely new tincture (**Tincture 1**), giving Figure 12.

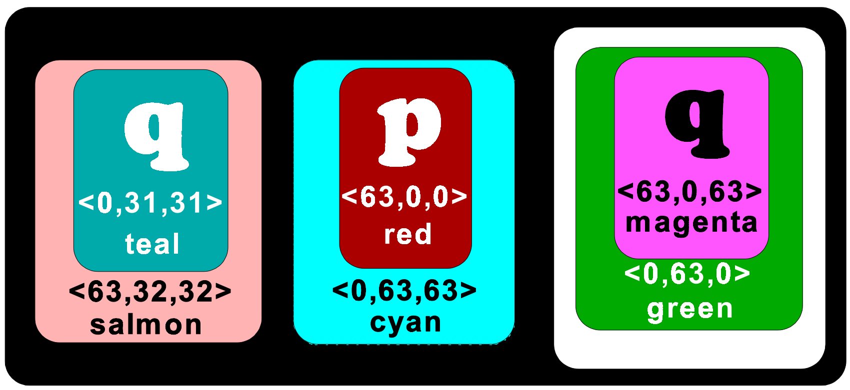

Figure 12

We may in turn iterate the partial graph containing ***p*** in the same manner, but to a different area, giving Figure 13. The new tincture chosen for the iterated instance is **Tincture 4**, which, I note, is accessible to **Tincture 3** (see Figure 1) by the concept of accessibility we have discussed.

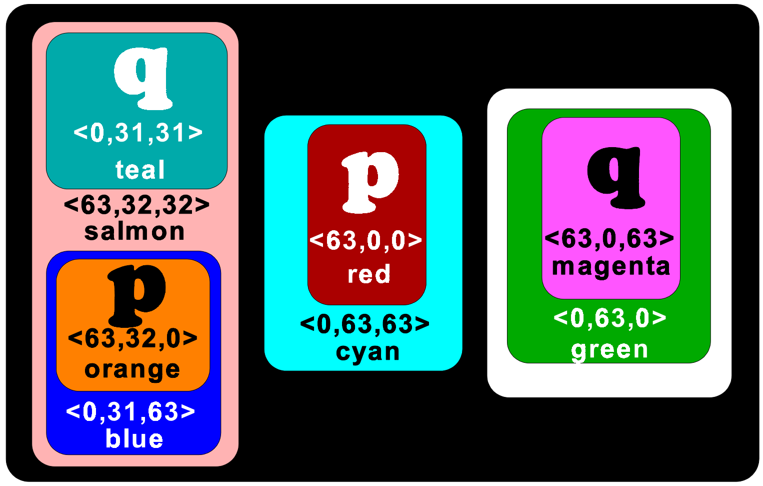

Figure 13

So far the iterations follow, as we have noted, the reasoning leading to Figure 8. But now a new twist appears. The just-iterated partial graph (which is of **Tincture 4**; as indicated in Figure 1, **Tincture 4** is *accessible to* its enclosing **Tincture 3**) is now entirely within an area of **Tincture 3**. The outermost area of the just-iterated graph is *oddly enclosed* in the overall graph. **Tincture 1**, thought of as a **LI**, may be extended inward by Beta-iteration, and then, by Beta-join-in-odd, connected to the **LI** which is **Tincture 4** in this graph; these tinctures are thereby identified, as in Figure 14.

Following the above reasoning, we may state tincture rules corresponding to the beta rules for insertion in odd and erasure in even

Ins$_{\gamma}$: A tincture **t** on any *verso*, along with the same tincture in all areas within the enclosure of that verso, may be identified with a tincture *which accesses* **t** and which ***totally surrounds*** **t**.

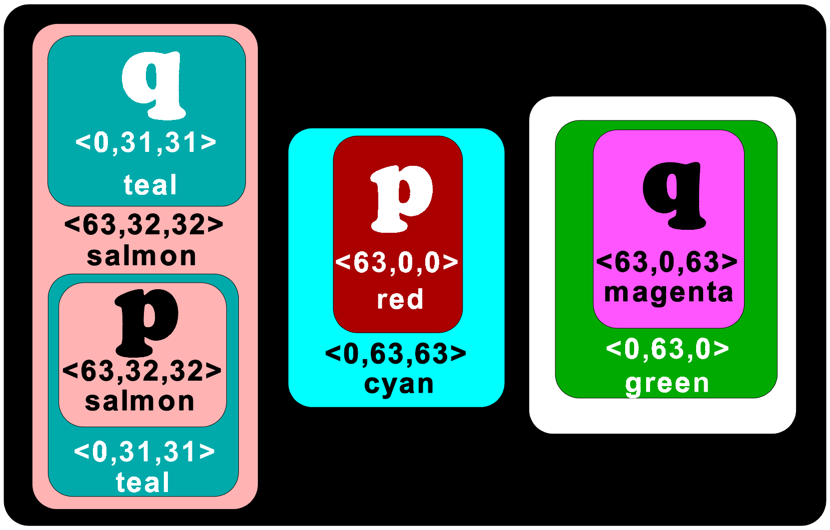

Figure 14

Ers$_\gamma$: A tincture **t** on any *recto*, along with the same tincture in all areas within the enclosure of that *recto*, may be replaced by a tincture *accessible to* **t** and entirely new to the whole graph.

We might further alter the graph of Figure 14 by applying Negative Double Cut in **Tincture 1** and Positive Double Cut in **Tincture 0**; this gives Figure 15.

This is the graphical equivalent of

(6) $$\boldsymbol{L(p \supset q) \supset (Lp \supset Lq)}$$

In similar fashion, we might have found graphical equivalents of formulas such as

(7) $$\boldsymbol{L(p \wedge q) \supset (Lp \wedge Lq)}$$

and

(8) $$\boldsymbol{(Lp \wedge Lq) \supset L(p \wedge q)}$$

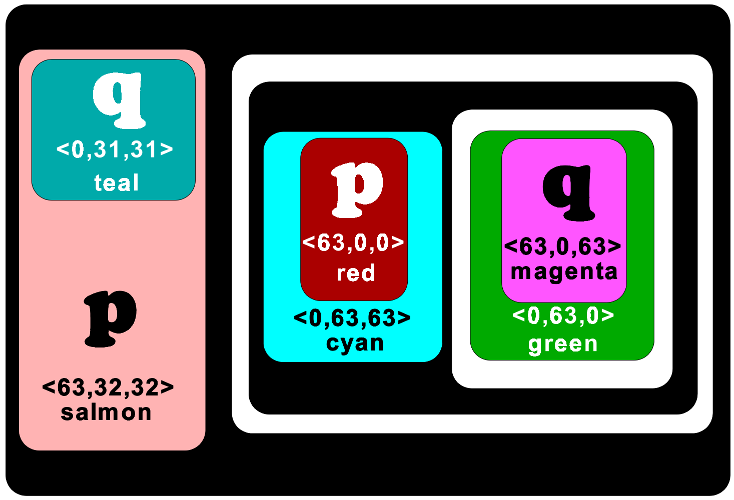

Figure 15

Note that with the rules as we have them, if a graph $\mathbf{A}$ is constructible on **Tincture 0**, then it will be constructible on any *recto*. Now, although other agreements might be arrived at between *Graphist* and *Interpreter* (4.552), one reasonable such agreement, and the one that we will take as applying to the tinctured graphs with which we work here, is, then, that all logical transformations which can be applied in "*Argent*" may be applied in all the colors to which *Argent* has access. Thus, where $\vdash \mathbf{A} @ \boldsymbol{w}$ means that $\mathbf{A}$ holds at $\boldsymbol{w}$ by virtue of the rules of transformation for ***EG***, we will have, for $\boldsymbol{a}$ the *Argent* world

(9) $$\text{If} \vdash \mathbf{A} @ \boldsymbol{a} \text{ then} \vdash \forall \boldsymbol{w}(\boldsymbol{Raw} \supset \mathbf{A} @ \boldsymbol{w})$$

which means that we have the equivalent of **necessitation**:

(10) $$\text{If} \vdash \alpha, \text{ then} \vdash \boldsymbol{L}\alpha$$

With this, and with (6), we then also have the equivalent of

(11) $$\text{If} \vdash \alpha \supset \beta \text{ then} \vdash \boldsymbol{L}\alpha \supset \boldsymbol{L}\beta$$

Expressions (6)-(11) mean that the logic we are dealing with contains the equivalent of Segerberg's (1971) Normal modal logic **K**, or, equivalently, Sobocinski's T° (see Zeman 1973).

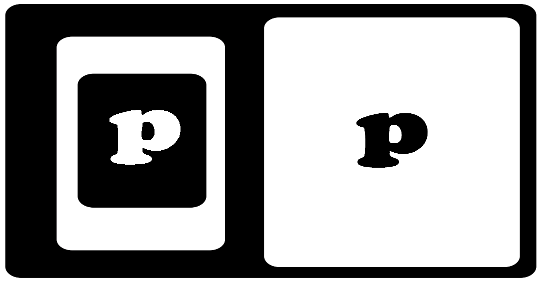

Figure 16

In addition, Figure 16 will result from the alpha rules alone. Then, by $\mathbf{Ers}_\gamma$ applied to the left (twice-enclosed) white recto, we get Figure 17 (the change in the *recto* takes its enclosed *verso* along with it from **Tincture 0** to **Tincture 1**).

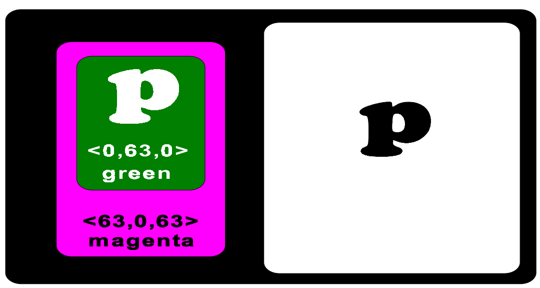

Figure 17

Since **Tincture 1** is accessible to **Tincture 0**, Figure 17 is the graphical equivalent of

(12) $$\boldsymbol{Lp \supset p}$$

The logic we are dealing with then contains the equivalent of the modal system T.

We may carry this a bit further; if we take the graph of Figure 10 and first perform an application of $\mathbf{Ins}_\alpha$ to get the partial graph in **Tincture 1**, and then perform $\mathbf{Itr}_\gamma$ to get the replica of that partial graph (in **Tincture 3**) in the inner *recto* area, we have Figure 18.

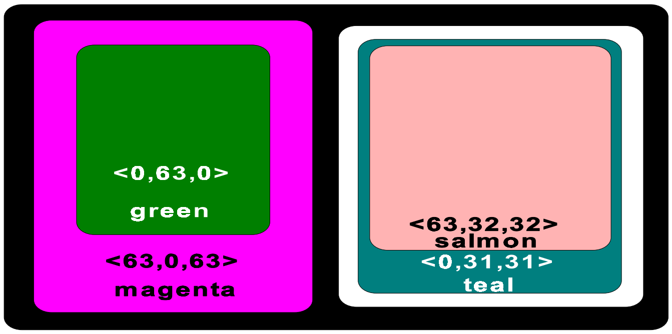

Figure 18

This figure may be further modified by an application of $\mathbf{Ins}_{\alpha}$, giving us Figure 19.

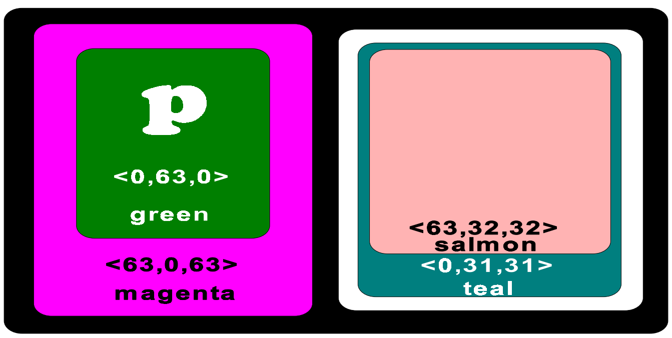

Figure 19

And by an application of $\mathbf{Itr}_{\gamma}$, this becomes Figure 20.

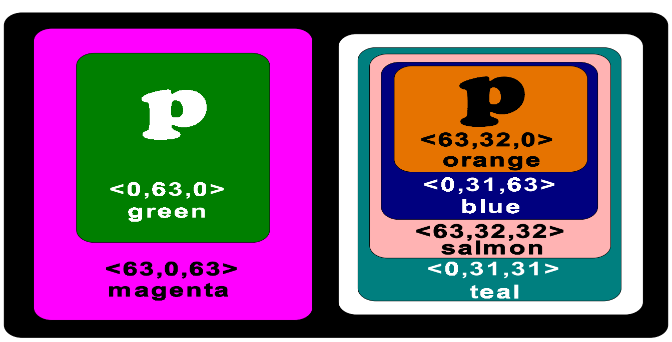

Figure 20

Given the conventions we have adopted regarding tincture and access, access is a transitive relation; the tincture of the just-iterated partial graph (which is accessible to the area onto which it is iterated, by $\mathbf{Itr}_\gamma$), is accessible to the tincture on which *that* tincture stands, and so the graph of Figure 20 is the equivalent of

(12) $$\boldsymbol{Lp \supset LLp}$$

and the graphical system we are dealing with includes the equivalent of the Lewis-modal S4.

So we have the basis of modal logic within the tinctures of ***PAP***. It is clear that many variations could be produced by altering the very natural access between worlds we have used; if we restrict ourselves to shades of grey (where $\mathbf{r} = \mathbf{g} = \mathbf{b}$, for example, we get a version of Prior's Diodorean modal logic (see Zeman 1973, 229 ff.). There is much to be examined here—where, for instance, do the "furs" of ***PAP*** fit in? But for now, we have presented an exercise in experimentation on diagrams which is, I believe, faithful to Peirce's own approach.

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APPENDIX

RULES FOR ALPHA AND BETA GRAPHS

Ins$_\alpha$: On any *verso* (area enclosed by an odd number of cuts) any graph may be scribed (*Rule of **Insertion in Odd***).

Ers$_\alpha$: On any *recto* (area enclosed by an even number, including 0, of cuts), any graph may be erased (*Rule of **Erasure in Even***).

Itr$_\alpha$: Any graph ***g*** may be iterated (scribed again) in the same area or inward, that is, into any area enclosed by at least all the cuts that enclose the original instance of ***g*** (*Rule of **Iteration***).

Dit$_\alpha$: Any partial graph which might have resulted from **Itr$_\alpha$** may be erased, regardless of how it is enclosed (*Rule of **Deiteration***).

Dc+$_\alpha$: A pair of cuts with nothing between them but blank **SA** may be scribed around any graph (*Rule of **Positive Double Cut***).

Dc-$_\alpha$: A pair of cuts with nothing between them but blank **SA** may be erased from around any graph (*Rule of **Negative Double Cut***).

In Addition to the Above, the Following Hold for Beta:

Ins$_\beta$: On any *verso*, **LI**'s may be joined at will.

Ers$_\beta$: On any *recto*, **LI**'s may be severed at will.

Itr$_\beta$: If ***g*** is iterated by **Itr$_\alpha$** and ***g*** has **LI**'s with parts unenclosed in its original area, those **LI** parts may be connected by **LI** to the corresponding parts of the iterated replica of ***g***. As a specific subcase of this rule, the end of an **LI** unenclosed in ***g*** may be extended to any area to which ***g*** might be iterated. Also, a “point of teridentity”--a branch--may be extended from any point on a **LI**.

Dit$_\beta$: Any partial graph which might have resulted from **Itr$_\beta$** may be “undone.”

Dc+$_\beta$: A pair of cuts with nothing between them but blank **SA** and **LI**'s which pass from completely without the outer cut to completely within the inner may be scribed around any graph.

Dc-$_\beta$: A pair of cuts with nothing between them but blank **SA** and **LI**'s which pass from completely without the outer cut to completely within the inner may be erased from around any graph.

RULES FOR TINCTURES

Ins$_\alpha$ and **Ers**$_\alpha$ apply unchanged to Tinctured EG; the tincture of the *recto* or *verso* involved does not matter. **Itr**$_\alpha$ and **Dit**$_\alpha$ are subcases of the corresponding γ-rules (specifically, they apply when there are no "isolated tinctures" in the partial graph to be iterated). Similarly, the Double-Cut rules for Alpha are the special cases of the corresponding γ-rules. We tentatively will take the Beta rules to hold as stated, observing, of course, any tincture-connected provisos associated with the Iteration-Deiteration and Double-Cut rules.

A tincture **t** is *isolated in* a (partial) graph ***g*** iff there is no occurrence of **t** in the whole graph outside of ***g***.

Although the meaning of *access* among tinctures might be developed in a variety of directions, for present purposes, let us first suppose that we have two tinctures (worlds), ***u*** and ***v***, associated respectively with the *recto-verso* pairs $<\boldsymbol{a},\boldsymbol{b}>$ and $<\boldsymbol{c},\boldsymbol{d}>$. With color ***h*** (in terms of red, green, blue) as $<\mathbf{r}_h,\mathbf{g}_h,\mathbf{b}_h>$, we say that (colors)

$$\boldsymbol{e} \le \boldsymbol{f} \text{ iff } \mathbf{r}_e \le \mathbf{r}_f \textit{ and } \mathbf{g}_e \le \mathbf{g}_f \textit{ and } \mathbf{b}_e \le \mathbf{b}_f;$$

we might say in this case that ***e*** is a *subcolor* of ***f***. Then (with ***u*** as $<\boldsymbol{a},\boldsymbol{b}>$ and ***u*** as $<\boldsymbol{c},\boldsymbol{d}>$),

$$\boldsymbol{v} \textit{ has access to } \boldsymbol{u} \text{ iff } \boldsymbol{c} \le \boldsymbol{a}$$

(equivalently, $\boldsymbol{b} \le \boldsymbol{d}$, by the relationship between colors representing *rectos* and *versos*)

Ins$_\gamma$: A tincture **t** on any *verso*, along with the same tincture in all areas within the enclosure of that *verso*, may be identified with a tincture *which acceses* **t** and which ***totally surrounds*** **t**.

Ers$_\gamma$: A tincture **t** on any *recto*, along with the same tincture in all areas within the enclosure of that *recto*, may be replaced by a tincture *acessible to* **t** and entirely new to the whole graph.

Itr$_\gamma$: A partial graph may be iterated in the same area or inward. *Isolated tinctures* in the original will be changed in the copy to *new tinctures accessible to the tincture of the area to which the iteration is performed*.

Dit$_\gamma$: A partial graph which might have resulted from **Itr**$_\gamma$ may be deiterated.

Dc+$_\gamma$: A pair of cuts with nothing between them but blank **SA**, and with the same tincture outside the outer cut, between the two cuts, and within the inner may be scribed around any graph.

Dc-$_\gamma$: A pair of cuts with nothing between them but blank **SA**, and with the same tincture outside the outer cut, between the two cuts, and within the inner may be erased from around any graph.

NOTES

1.As is common in Peirce scholarship, citations from Peirce 1936-58 will be indicated by volume-point-paragraph number; thus 4.530 is paragraph 530 of volume 4 of the ***Collected Papers***.

2.In 4.528, Peirce gives us notation for expressing some relationships of graphs (propositions) to sheets of assertion (universes of discourse).

3.“@” may be taken as a relation between propositions and “possible worlds”; our intuitive reading of

$$\alpha \mathbin{@} \boldsymbol{b}$$

would be “proposition α at [holds at, is true at] ‘world’ ***b***.” @ will, of course, have the following properties:

Where ***p*** is any “atomic” formula, ***p*** @ ***b*** iff ***p*** is assigned true at “world” ***b***

$(\alpha \wedge \beta) \mathbin{@} \boldsymbol{b}$ iff $(\alpha \mathbin{@} \boldsymbol{b}) \wedge (\beta \mathbin{@} \boldsymbol{b})$;

$({\sim}\alpha) \mathbin{@} \boldsymbol{b}$ iff ${\sim}(\alpha \mathbin{@} \boldsymbol{b})$.

So far as quantifiers are concerned,

$(\forall \boldsymbol{x}\boldsymbol{B}\boldsymbol{x}) \mathbin{@} \boldsymbol{b}$ iff $\forall \boldsymbol{x}(\boldsymbol{B}\boldsymbol{x} \mathbin{@} \boldsymbol{b})$.

4.We shall shortly explain these terms.

5.“The whole of any continuous part of the exposed surface in one tincture shall be termed a Province” (4.553).

6.The correlation between *color index* and *color* is arbitrary or, we might prefer to say, “user-defined.”

7.Limitations on the presentation of this material dictate that we tag the colors with identifiers to avoid having to visually discriminate between the shade of grey that represents blue, for example, and that which represents red. In the diagram of Figure 1, “**T*n*r**” designates the *recto* of **Tincture n**, while **T*n*v** designates its *verso*; the rest of the first line gives the VGA256 color index in hexadecimal; the “**0x**” prefix is a C-language convention for hexadecimal. If the colors are imagined as laid out on a 16×16 matrix with columns (counted from the left) mapping the first digit and rows (from the top) the second digit of each index, the colors will be laid out in approximately the pattern given in Figure 1. The second line in each box gives the

<**red, green,blue**>

proportions of each color, and the third line gives an intuitive color designation, for reference. These indicators will appear in Helvetica bold (black or white):

blue yellow red

These names are in no way part of the graphs in which they appear, but are for convenience in recognizing the tinctures involved. The letters representing graphs will be considerably larger, and ordinarily in Cooper Black (which sometimes may be white; in tinctured graphs, lettering on *verso* will generally be white, and on *recto*, lettering will be black):

p q r A B

8. We shall shortly give meaning to this word.

9. Which is connected with the "Normal" modal logics of Segerberg 1971, 74 ff.

10. This tinctures's *recto* color is <63,63,63>—white—the fullest saturation of red, green and blue available in VGA256. Any other color will be included in white, and so Tincture 0 will have access to any of the other tinctures we discuss.

11. Again, presumably *argent*; appropriately, metallic silver appears sometimes white and sometimes black.

12. Which follows generally the suggestions of "An Improvement on the Gamma Graphs" (4.573 ff.). This involves, as we have noted, cutting down into the **SA** to find other levels of possibility. **PAP** suggests a somewhat different picture, in which the base tincture is defined by the Province at the "March" of the graph (4.553), and Provinces in other tinctures are thought of as overlying that base and each other (and so being accessible, perhaps, directly rather than only through cuts—Peirce does not give much detail on this) (Ibid.).

13. I am making an effort in this to stay as close as possible to Peirce's own uses of terms such as "indefinite." For further remarks on Peirce's logic of the indeterminate, see Zeman 1988.

14.I admit here to using the expression "Line of Identity" where Peirce might have preferred "Ligature." I don't think that any confusion will result.